Pergamon

Appl. Math. Lett. Vol. 10, No. 3, pp. 89–94, 1997

0893-9659/97 $7.00 + 0.00

The One-Dimensional Heat Equation with a Nonlocal Initial Condition

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(Received June 1996; accepted September 1996)

Communicated by B. J. Matkowsky

Abstract—A boundary value problem for the one-dimensional heat equation is considered under the constraint of a nonlocal initial condition. The problem is investigated by conversion to a Fredholm integral equation. The solution of the integral equation is derived as an eigenfunction expansion that is valid for situations in which either uniqueness or nonuniqueness applies. Circumstances under which a resonance effect occurs are discussed and illustrated with an example.

1. INTRODUCTION

We consider the one-dimensional heat equation

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = q(x,t), \qquad 0 < x < \ell, \quad t > 0, \tag{1}$$

subject to the boundary conditions

$$\frac{\partial v}{\partial x}(0,t) = \alpha_0 v(0,t), \quad \frac{\partial v}{\partial x}(\ell,t) = -\alpha_\ell v(\ell,t), \qquad t > 0, \tag{2}$$

where $\alpha_0 > 0$, $\alpha_\ell > 0$. In place of the classical specification of initial data, we impose the nonlocal initial condition

$$v(x,0) = g(x) + u(x), \qquad u(x) \equiv \int_0^\infty h(x,t)v(x,t)\,dt. \tag{3}$$

In this problem, $q(x,t)$, $h(x,t)$, and $g(x)$ are given functions, continuous in their arguments.

It is clear that (1)–(3) constitutes a nonstandard problem since the initial data $v(x,0)$ is specified in terms of the solution over all $t \ge 0$. There are a variety of physical situations that might be modeled by (1)–(3). A straightforward interpretation is that $v(x,t)$ represents the concentration of some diffusing substance, whose initial level is unspecified, although it is known that this initial level must be balanced against some weighted average of all future levels of concentration.

Research supported by NSF under Grants DMS-9401016 and DMS-9510082.

Typeset by $\mathcal{AMS}$-TEX

A recent paper [1] considered the existence, uniqueness, and long term decay of a solution to (1)–(3) in a more general setting, which also allowed for a nonlinear source $q = q(x,t,v)$. That investigation proposed the nonlocal initial condition (3) as a generalization of a discrete version of this condition treated in [2,3]. The results of [1–3] all pertain to the case of a unique solution, and appropriate constraints are imposed on $h(x,t)$, or its discrete equivalent, to insure uniqueness.

Our intention here is to initiate an investigation which will focus on the broader implications of the nonlocal initial condition (3) relative to nonuniqueness and nonsolvability, as well as to unique solvability. To demonstrate the basic aspects of these issues in a relatively explicit manner, we will confine our attention to the linear, one-dimensional heat equation (1) with unmixed boundary conditions (2). For convenience, we will also assume that the weight function $h(x,t)$ is such that

$$\begin{gathered} h(x,t) = r(x)w(t), \qquad 0 < r(x) \le 1, \quad |w(t)| \le M, \\ 0 < \left| \int_0^\infty w(t) e^{-\lambda t}\, dt \right| < \infty, \qquad \lambda > 0. \end{gathered} \tag{4}$$

Our analysis will reduce (1)–(3) to an integral equation for $u(x)$. We can then focus on the determination of $u(x)$, because, once it is known, then (1)–(3) represents a standard problem for the heat equation.

We begin by expressing (1)–(3) in the equivalent form

$$v(x,t) = F(x,t) + \int_0^\ell G\,(x,t \mid \xi, 0)\, u(\xi)\, d\xi, \qquad 0 \le x \le \ell, \quad t \ge 0, \tag{5}$$

where

$$F(x,t) \equiv \int_0^t \int_0^\ell G\,(x,t \mid \xi, s)\, q(\xi,s)\, d\xi\, ds + \int_0^\ell G\,(x,t \mid \xi, 0)\, g(\xi)\, d\xi. \tag{6}$$

Here $G(x,t \mid \xi, s)$ is the Green's function associated with the linear heat operator subject to the boundary conditions (2). It can be expressed as

$$G\,(x,t \mid \xi, s) = H(t-s) \sum_{n=1}^\infty \varphi_n(\xi)\varphi_n(x) e^{-\lambda_n (t-s)}, \tag{7}$$

where (φ_n, λ_n), $n = 1, 2, \ldots$, satisfy the eigenvalue problem

$$\begin{gathered} \varphi_n''(x) + \lambda_n \varphi_n(x) = 0, \qquad 0 < x < \ell, \\ \varphi_n'(0) = \alpha_0 \varphi_n(0), \qquad \varphi_n'(\ell) = -\alpha_\ell \varphi_n(\ell), \end{gathered} \tag{8}$$

with the orthonormalization condition

$$\int_0^\ell \varphi_n(x)\varphi_m(x)\, dx = \delta_{mn}. \tag{9}$$

Upon multiplication of (5) by $h(x,t)$ and integration over $(0,\ell)$, we obtain a Fredholm integral equation of the second kind for $u(x)$,

$$\tilde{K}u(x) \equiv \int_0^\ell \tilde{k}(x,\xi) u(\xi)\, d\xi = u(x) - \tilde{F}(x), \qquad 0 \le x \le \ell, \tag{10}$$

where

$$\tilde{F}(x) \equiv \int_0^\infty h(x,t) F(x,t)\, dt \tag{11}$$

and

$$\tilde{k}(x,\xi) \equiv \int_0^\infty h(x,t) G(x,t \mid \xi, 0)\ dt. \tag{12}$$

2. ANALYSIS OF THE INTEGRAL EQUATION

With $h(x,t)$ satisfying (4), we can convert the integral equation (10) to a symmetric form by defining

$$U(x) \equiv [r(x)]^{-1/2}\, u(x). \tag{13}$$

This yields

$$KU(x) \equiv \int_0^\ell k(x,\xi) U(\xi)\, d\xi = U(x) - f(x), \qquad 0 \le x \le \ell, \tag{14}$$

where

$$f(x) \equiv [r(x)]^{1/2} \int_0^\infty w(t) F(x,t)\, dt \tag{15}$$

and

$$k(x,\xi) \equiv [r(x) r(\xi)]^{1/2} \int_0^\infty w(t) G(x,t \mid \xi, 0)\ dt. \tag{16}$$

The symmetry property of the kernel $k(x,\xi) = k(\xi,x)$ is readily seen from (7), which allows (16) to be expressed as

$$k(x,\xi) = [r(x) r(\xi)]^{1/2} \sum_{n=1}^\infty \hat{w}(\lambda_n) \varphi_n(\xi) \varphi_n(x), \tag{17}$$

where

$$\hat{w}(\lambda_n) = \int_0^\infty w(t) e^{-\lambda_n t}\, dt, \qquad n = 1, 2 \dots. \tag{18}$$

Since the eigenvalues λ_n for (8) are real and positive, then $\hat{w}(\lambda_n)$ can be interpreted as the Laplace transform of $w(t)$ evaluated at the eigenvalues. Moreover, the bounds in (4) provide that

$$0 < |\hat{w}(\lambda_n)| \le \int_0^\infty |w(t)|\, e^{-\lambda_n t}\, dt \le \frac{M}{\lambda_n}, \qquad n = 1, 2 \dots. \tag{19}$$

To see that $k(x,\xi)$ is a Hilbert-Schmidt kernel, we have that

$$\begin{aligned} \int_0^\ell \int_0^\ell k^2(x,\xi)\, dx\, d\xi &= \int_0^\ell \int_0^\ell r(x) r(\xi) \left[\sum_{n=1}^\infty \hat{w}(\lambda_n) \varphi_n(\xi) \varphi_n(x) \right]^2 d\xi\, dx \\ &\le \sum_{n=1}^\infty [\hat{w}(\lambda_n)]^2 \le M^2 \sum_{n=1}^\infty \left(\frac{1}{\lambda_n} \right)^2 < \infty. \end{aligned} \tag{20}$$

The finiteness of the upper bound in (20) follows from the fact that the eigenvalues for (8) are such that $\lambda_n = 0(n^2)$ as $n \to \infty$.

To solve (14), we will utilize the eigenfunctions Φ_n, $n = 1, 2, \dots$, which satisfy the eigenvalue problem

$$K\Phi_n(x) \equiv \int_0^\ell k(x,\xi) \Phi_n(\xi)\, d\xi = \Lambda_n \Phi_n(x), \qquad 0 \le x \le \ell. \tag{21}$$

For symmetric Hilbert-Schmidt kernels, it is well known (e.g., [4,5]) that (21) has real eigenvalues Λ_n and an orthonormal set of eigenfunctions

$$\int_0^\ell \Phi_m(x) \Phi_n(x)\, dz = \delta_{mn}. \tag{22}$$

To establish that there are an infinite number of $\Phi_n(x)$ which form a complete set, it is only necessary to show that $\Lambda_n \neq 0$, $n = 1, 2, \ldots$. If this were not true, then there would be a nontrivial solution of

$$\int_0^\ell \sum_{n=1}^\infty \hat{g}(\lambda_n)\varphi_n(\xi)\varphi_n(x)\,[r(\xi)]^{1/2}\,\Phi(\xi)\,d\xi = 0, \qquad 0 \le x \le \ell. \tag{23}$$

Since the $\varphi_n(x)$ do comprise a complete set, we can express any solution of (23) as an expansion of the form

$$\Phi(x) = [r(x)]^{-1/2} \sum_{n=1}^\infty c_n \varphi_n(x). \tag{24}$$

Substitution of (24) into (23) yields

$$\sum_{n=1}^\infty \hat{w}(\lambda_n) c_n \varphi_n(x) = 0, \tag{25}$$

which implies that $\hat{w}(\lambda_n)c_n = 0$, and hence, $c_n = 0$, $n = 1, 2, \ldots$. Thus, $\Phi\ (x) \equiv 0$ is the only solution of (23), which thereby excludes zero as an eigenvalue for (21). Moreover, there will be an infinite number of eigenvalues with $\Lambda_n \to 0$ as $n \to \infty$.

It is also of interest to establish conditions under which the integral operator K, defined by (14), is positive. Using the expansion (24) for an arbitrary $\Phi\ (x)$, it follows that

$$\begin{aligned}\int_0^\ell \Phi(x) K \Phi(x)\,dx &= \int_0^\ell \int_0^\ell \sum_{n=1}^\infty \hat{w}(\lambda_n)\varphi_n(\xi)\varphi_n(x)\,[r(x)r(\xi)]^{1/2}\,\Phi(x)\Phi(\xi)\,d\xi\,dx \\ &= \sum_{n=1}^\infty \hat{w}(\lambda_n)\left[\int_0^\ell \varphi_n(\xi) \sum_{m=1}^\infty c_m \varphi_m(\xi)\,d\xi\right]^2 = \sum_{n=1}^\infty \hat{w}(\lambda_n) c_n^2.\end{aligned} \tag{26}$$

From (26) it is clear that if $\hat{w}(\lambda_n) > 0$, $n = 1, 2, \ldots$, then K is a positive operator, and consequently, $\Lambda_n > 0$, $n = 1, 2, \ldots$.

We now consider the construction of a solution to the integral equation (14). Two approaches will be presented, namely, the Neumann series and an eigenfunction expansion.

For a Hilbert-Schmidt kernel, it is well known (e.g., [4,5]) that the Neumann series

$$U(x) = f(x) + \sum_{n=1}^\infty K^n f(x) \tag{27}$$

converges to the unique solution of (14) whenever

$$|\Lambda_1| < 1. \tag{28}$$

In (27), K^n stands for the n^{th} iteration of the integral operator K. We can replace the optimal condition (28) by a sufficient condition for convergence of the Neumann series, which utilizes the fact that $|\Lambda_1|$ is bounded above by the square root of the integral in (20). Thus, for the convergence of (27), it is sufficient that

$$M\left[\sum_{n=1}^\infty \left(\frac{1}{\lambda_n}\right)^2\right]^{1/2} < 1. \tag{29}$$

The limitation of the Neumann series is that it cannot provide any information about solutions of (14) associated with $|\Lambda_1| \ge 1$. To address this issue, we consider an eigenfunction expansion.

Since the $\Phi_n(x)$ have been established as a complete orthonormal set, we can seek to solve (14) by means of the expansion

$$U(x) = \sum_{n=1}^{\infty} a_n \Phi_n(x). \tag{30}$$

Substitution into (14) and utilizing (21) yields

$$\sum_{n=1}^{\infty} (1 - \Lambda_n)\, a_n \Phi_n(x) = f(x). \tag{31}$$

By the orthonormality property (22), it follows that

$$(1 - \Lambda_n)\, a_n = \int_0^{\ell} f(x)\Phi_n(x)\, dx, \qquad n = 1, 2, \dots. \tag{32}$$

The alternative possibilities of (32) are clear. If $\Lambda_n \neq 1$, $n = 1, 2, \dots$, then each a_n is explicitly determined from (32). This yields the unique solution of (14) as

$$U(x) = \sum_{n=1}^{\infty} \left[\frac{\int_0^{\ell} f(\xi)\Phi_n(\xi)\, d\xi}{1 - \Lambda_n} \right] \Phi_n(x). \tag{33}$$

Alternatively, if $\Lambda_N = 1$ for some specific $n = N$, then (32) implies that (14) has no solution unless

$$\int_0^{\ell} f(\xi)\Phi_N(\xi)\, d\xi = 0. \tag{34}$$

Whenever (34) is satisfied, then (14) has the nonunique solution

$$U(x) = a_N \Phi_N(x) + \sum_{\substack{n=1 \\ n \neq N}}^{\infty} \left[\frac{\int_0^{\ell} f(\xi)\Phi_n(\xi)\, d\xi}{1 - \Lambda_n} \right] \Phi_n(x), \tag{35}$$

where a_N is arbitrary.

To summarize our results, we have that (1)–(3) becomes a standard problem for the heat equation, whose solution is given by (5), once $u(x)$ has been determined. The determination of $u(x)$ follows from (13) and the solution of (14) for $U(x)$. The unique solution of (14) is given by (27) whenever (28) or (29) is fulfilled, and more generally by (33) whenever $\Lambda_n \neq 1$, $n = 1, 2, \dots$. If there is some $\Lambda_N = 1$, then there is no solution unless (34) is satisfied, whereupon there is a nonunique solution given by (35).

The form of (33) suggests that the solution of (1)–(3) can exhibit a type of resonance effect. This *initial data resonance* arises when $h(x, t)$ is "tuned" in a manner that moves one of the Λ_n near to unity. That is, if there is some $n = N$ such that $\Lambda_N = 1 - \epsilon$, $0 < |\epsilon| \ll 1$, then $U(x)$ will be strongly dominated by the mode associated with $\Phi_N(x)$.

3. SPECIAL CASE OF TEMPORAL WEIGHTING

The situation in which the weight function $h(x, t)$ has only temporal dependence yields a considerable simplification of the results in Section 2. We consider the special case of (4) in which

$$h(x, t) = w(t), \qquad 0 \leq w(t) \leq M. \tag{36}$$

For this case the eigenvalue problem (21) takes the form

$$\int_0^{\ell} \left[\sum_{m=1}^{\infty} \hat{w}(\lambda_m)\varphi_m(x)\varphi_m(\xi) \right] \Phi_n(\xi)\, d\xi = \Lambda_n \Phi_n(x), \qquad 0 \leq x \leq \ell. \tag{37}$$

It is easily verified that (37) is satisfied by

$$\Phi_n(x) = \varphi_n(x), \qquad \Lambda_n = \hat{w}(\lambda_n), \quad n = 1, 2, \ldots. \tag{38}$$

In view of (36), it follows from (18) that $\hat{w}(\lambda_n) > 0$ and

$$\hat{w}(\lambda_1) > \hat{w}(\lambda_2) > \cdots > \hat{w}(\lambda_n) \to 0, \qquad \text{as } n \to \infty. \tag{39}$$

Thus, the Neumann series (27) converges to the unique solution of (14) with $u(x) = U(x)$ whenever $\hat{w}(\lambda_1) < 1$. In terms of the eigenfunction expansion of the solution to (14), if $\hat{w}(\lambda_n) \neq 1$, $n = 1, 2, \ldots$, then

$$u(x) = U(x) = \sum_{n=1}^{\infty} \left[\frac{\int_0^\ell f(\xi)\varphi_n(\xi)\, d\xi}{1 - \hat{w}(\lambda_n)} \right] \varphi_n(x) \tag{40}$$

provides the unique solution of (14). If $\hat{\omega}\ (\lambda_N) = 1$ for some specific $n = N$, then the expansion

$$u(x) = U(x) = a_N \varphi_N(x) + \sum_{\substack{n=1 \\ n \neq N}}^{\infty} \left[\frac{\int_0^\ell f(\xi)\varphi_n(\xi)\, d\xi}{1 - \hat{w}(\lambda_n)} \right] \varphi_n(x), \tag{41}$$

with a_N arbitrary, provides the nonunique solution of (14) if and only if

$$\int_0^\ell f(\xi)\varphi_N(\xi)\, d\xi = 0. \tag{42}$$

Thus, for the special case of (36), $u(x)$ is given for all situations in which it exists by either (40) or (41). These expansions are in terms of the eigenfunctions of the Sturm-Liouville problem (8). With $u(x)$ thus determined, then (5) provides the (unique or nonunique) solution of (1)–(3).

As an example of this special case that illustrates the effect of initial data resonance, we consider (1)–(3) with

$$q(x,t) = 0, \qquad g(x) = 1, \qquad \alpha_0 = \alpha_\ell = \infty, \qquad \ell = \pi, \tag{43}$$

and the weight function given by

$$h(x,t) = \beta e^{-\epsilon t}, \qquad \beta > 0, \quad 0 < \epsilon \ll 1. \tag{44}$$

It follows that

$$v(x,0) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{n^2 + \epsilon}{n^2 + \epsilon - \beta} \right] \left[\frac{1 - (-1)^n}{n} \right] \sin nx, \qquad 0 < x < \pi. \tag{45}$$

With ϵ fixed in (45), we find that β can be used as a "tuning" parameter to cause a particular mode to dominate the initial data. That is, by setting $\beta = N^2$, $N = 1, 3, 5, \ldots$, then the N^{th} mode is scaled by $4N/\pi\epsilon$, and hence, will dominate in (45).

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