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0521794870 - An Introduction to Magnetohydrodynamics

P. A. Davidson

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Magnetic fields influence many natural and man-made flows. They are routinely used in industry to heat, pump, stir and levitate liquid metals. There is the terrestrial magnetic field which is maintained by fluid motion in the earth's core, the solar magnetic field which generates sunspots and solar flares, and the galactic field which influences the formation of stars. This is an introductory text on magnetohydrodynamics (MHD) – the study of the interaction of magnetic fields and conducting fluids.

This book is intended to serve as an introductory text for advanced undergraduate and postgraduate students in physics, applied mathematics and engineering. The material in the text is heavily weighted towards incompressible flows and to terrestrial (as distinct from astrophysical) applications. The final sections of the text also contain an outline of the latest advances in the metallurgical applications of MHD and so are relevant to professional researchers in applied mathematics, engineering and metallurgy.

Dr. P.A. Davidson is a Reader in Fluid Mechanics at the University of Cambridge, where his current research is in fluid mechanics in process metallurgy, turbulence and stability theory. He is the author of over 50 publications, and was awarded the Institute of Materials prize in 1996 for the best paper on non-ferrous metallurgy.

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P. A. DAVIDSON
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For my family

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Preface

Prefaces are rarely inspiring and, one suspects, seldom read. They generally consist of a dry, factual account of the content of the book, its intended readership and the names of those who assisted in its preparation. There are, of course, exceptions, of which Den Hartog's preface to a text on mechanics is amongst the wittiest. Musing whimsically on the futility of prefaces in general, and on the inevitable demise of those who, like Heaviside, use them to settle old scores, Den Hartog's preface contains barely a single relevant fact. Only in the final paragraph does he touch on more conventional matters with the observation that he has 'placed no deliberate errors in the book, but he has lived long enough to be quite familiar with his own imperfections'.

We, for our part, shall stay with a more conventional format. This work is more of a text than a monograph. Part A (the larger part of the book) is intended to serve as an introductory text for (advanced) undergraduate and post-graduate students in physics, applied mathematics and engineering. Part B, on the other hand, is more of a research monograph and we hope that it will serve as a useful reference for professional researchers in industry and academia. We have at all times attempted to use the appropriate level of mathematics required to expose the underlying phenomena. Too much mathematics can, in our opinion, obscure the interesting physics and needlessly frighten the student. Conversely, a studious avoidance of mathematics inevitably limits the degree to which the phenomena can be adequately explained.

It is our observation that physics graduates are often well versed in the use of Maxwell's equations, but have only a passing acquaintance with fluid mechanics. Engineering graduates often have the opposite background. Consequently, we have decided to develop, more or less from first principles, those aspects of electromagnetism and fluid mechanics

which are most relevant to our subject, and which are often treated inadequately in elementary courses.

The material in the text is heavily weighted towards incompressible flows and to engineering (as distinct from astrophysical) applications. There are two reasons for this. The first is that there already exist several excellent texts on astrophysical, geophysical and plasma MHD, whereas texts oriented towards engineering applications are somewhat thinner on the ground. Second, in recent years we have witnessed a rapid growth in the application of MHD to metallurgical processes. This has spurred a great deal of fruitful research, much of which has yet to find its way into textbooks or monographs. It seems timely to summarise elements of this research. We have not tried to be exhaustive in our coverage of the metallurgical MHD, but we hope to have captured the key advances.

The author is indebted to the late D. Crighton, without whose support this text would never have seen the light of day, to H.K. Moffatt and J.C.R. Hunt for their constant advice over the years, to K. Graham for typing the manuscript, and to C. Davidson for her patience. Above all, the author would like to thank Stephen Davidson who painstakingly read each draft, querying every ambiguity and exposing the many inconsistencies in the original text.